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LETTER TO THE EDITOR

Surface terms in higher derivative gravity

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Abstract. Surface terms in a gravitational action containing the quadratic curvature scalars R^2, $R^{\alpha\beta}R_{\alpha\beta}$ and $R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}$ are considered and it is shown that a suitable surface term exists only for the combination $R^2-4R^{\alpha\beta}R_{\alpha\beta}+R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}$. The effect that the inclusion of this term in the action has on black hole evaporation is investigated.

One of the more striking results of quantum field theory in curved space–time is that the requirement of renormalisability demands that the classical gravitational action should be supplemented by terms quadratic in the curvature (Utiyama and De Witt 1962). Moreover, the inclusion of such terms leads to a renormalisable theory of gravity interacting with matter (Stelle 1977), although these terms also lead to negative probabilities and acausal behaviour and the correct physical interpretation of the theory, if any exists, is unclear. Recently, Gibbons and Hawking (1977) have argued that the Einstein gravitational action is incomplete and must be supplemented by a surface term involving the integral over the boundary of space–time of the trace of the second fundamental form. In this Letter, I investigate the surface terms corresponding to the quadratic curvature scalars R^2, $R^{\alpha\beta}R_{\alpha\beta}$, $R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}$ and find that a suitable surface term exists only for the particular linear combination $R^2-4R^{\alpha\beta}R_{\alpha\beta}+R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}$ which in four dimensions is the Gauss–Bonnet invariant. Hence if one accepts the necessity for surface terms in the action, renormalisability of quantum gravity cannot be achieved simply by including independent scalars R^2, $R^{\alpha\beta}R_{\alpha\beta}$ and $R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}$ in the action. I also show that such a term makes a very simple modification to the partition function of a black hole which suggests that the end state of black hole evaporation is a black hole of Planck dimensions, although this conclusion is rather tentative as the effect of the back reaction is not considered.

Let

$$I_1=\int R^2\sqrt{-g}\,\mathrm{d}^4x$$

$$I_2=\int R^{\alpha\beta}R_{\alpha\beta}\sqrt{-g}\,\mathrm{d}^4x$$

$$I_3=\int R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}\sqrt{-g}\,\mathrm{d}^4x.$$

Under variations in the metric, $\delta g_{\alpha\beta}$, which vanish on the boundary of the region of integration

$$\delta I_1 = \int [\tfrac{1}{2}(R^2 g^{\alpha\beta} - 4RR^{\alpha\beta}) + 2(R^{;\alpha\beta} - \Box R g^{\alpha\beta})]\, \delta g_{\alpha\beta}\sqrt{-g}\, \mathrm{d}^4x - 2\int Rh^{\alpha\beta}n^\rho \nabla_\rho\, \delta g_{\alpha\beta}\, \mathrm{d}\Sigma$$

$$\delta I_2 = \int [\tfrac{1}{2}R^{\mu\nu}R_{\mu\nu}g^{\alpha\beta} - 2R_{\mu\nu}R^{\mu\alpha\nu\beta} + R^{;\alpha\beta} - \Box R^{\alpha\beta} - \tfrac{1}{2}\Box R g^{\alpha\beta}]\sqrt{-g}\, \mathrm{d}^4x$$
$$+ \int [R_{\mu\nu}(h^{\mu\nu}h^{\alpha\beta} - h^{\mu\alpha}h^{\nu\beta}) - Rh^{\alpha\beta}]n^\rho\nabla_\rho\, \delta g_{\alpha\beta}\, \mathrm{d}\Sigma$$

$$\delta I_3 = \int [\tfrac{1}{2}R^{\lambda\mu\rho\sigma}R_{\lambda\mu\rho\sigma}g^{\alpha\beta} - 2R^{\alpha}{}_{\lambda\mu\rho}R^{\beta\lambda\mu\rho} - 4\Box R^{\alpha\beta} + 2R^{;\alpha\beta} + 4R^{\alpha\lambda}R^{\beta}_{\lambda}$$
$$- 4R_{\mu\nu}R^{\mu\alpha\nu\beta}]\, \delta g_{\alpha\beta}\sqrt{-g}\, \mathrm{d}^4x \mp 4\int R^{\lambda\alpha\mu\beta}n_\lambda n_\mu n^\rho\nabla_\rho\, \delta g_{\alpha\beta}\, \mathrm{d}\Sigma$$

where $h_{\mu\nu} = g_{\mu\nu} \mp n_\mu n_\nu$ is the induced metric on the boundary, n_μ is the normal to the boundary hypersurface normalised so that $n^\mu n_\mu = \pm 1$ with $+1$ for a time-like hypersurface, and $\mathrm{d}\Sigma$ is the covariant surface element $\sqrt{\pm h}\, \mathrm{d}^3x$ where $h = \det(h_{\mu\nu})$. Field equations based on an action containing I_1, I_2 and I_3 follow if one assumes that $n^\rho\nabla_\rho\, \delta g_{\alpha\beta}$ vanishes on the boundary. However, Gibbons and Hawking (1977) argue that this is too restrictive in the quantum theory and that the surface terms in δI_1, δI_2 and δI_3 should be cancelled by including appropriate surface terms in the action. Possible surface terms are

$$J_1 = \int K^{\alpha\beta}{}_{\|\alpha\beta}\, \mathrm{d}\Sigma \qquad J_2 = \int \Delta K\, \mathrm{d}\Sigma \qquad J_3 = \int RK\, \mathrm{d}\Sigma$$

$$J_4 = \int R_{\alpha\beta}h^{\alpha\beta}K\, \mathrm{d}\Sigma \qquad J_5 = \int R_{\alpha\beta}K^{\alpha\beta}\, \mathrm{d}\Sigma \qquad J_6 = \int R_{\alpha\beta\gamma\delta}K^{\alpha\gamma}h^{\beta\delta}\, \mathrm{d}\Sigma$$

$$J_7 = \int K^3\, \mathrm{d}\Sigma \qquad J_8 = \int KK^{\alpha\beta}K_{\alpha\beta}\, \mathrm{d}\Sigma \qquad J_9 = \int K^{\alpha\beta}K_{\beta\gamma}K^{\gamma}_{\alpha}\, \mathrm{d}\Sigma$$

where $K_{\alpha\beta}$ is the second fundamental form, K its trace, $\Delta K \equiv K_{\|\alpha}{}^{\alpha}$ and $\|$ denotes the covariant derivative based on the metric $h_{\alpha\beta}$. In addition to the terms above one might expect surface integrals of single derivatives of the Riemann curvature. Only derivatives within the surface are allowed and the only such term possible is $R_{\alpha\beta;\gamma}n^\alpha h^{\beta\gamma}$. However, the following identity, derived from the Codacci equation, shows that this term is not independent of those given above:

$$K^{\alpha\beta}{}_{\|\alpha\beta} - \Delta K = R_{\alpha\beta;\gamma}n^\alpha h^{\beta\gamma} + R_{\alpha\beta}K^{\alpha\beta} + KR_{\alpha\beta}h^{\alpha\beta} - RK.$$

Terms involving single derivatives of the extrinsic curvature $K_{\alpha\beta}$ are also related to those above, for example,

$$K^{\alpha\gamma}K_{\alpha\beta;\gamma}n^\beta = -K^{\alpha\gamma}K_{\alpha\beta}K^{\beta}_{\gamma}.$$

Under variations $\delta g_{\alpha\beta}$ which vanish on the boundary, both J_1 and J_2 yield third derivatives of $\delta g_{\alpha\beta}$ but the remaining terms yield only first and second derivatives of

$\delta g_{\alpha\beta}$. A straightforward calculation leads to

$$\delta J_1 = \frac{1}{2}\int h^{\alpha\mu} h^{\beta\nu} n^\rho \, \delta g_{\alpha\beta;\rho\mu\nu} \, d\Sigma + \ldots$$

$$\delta J_2 = \frac{1}{2}\int h^{\alpha\beta} h^{\mu\nu} n^\rho \, \delta g_{\alpha\beta;\rho\mu\nu} \, d\Sigma + \ldots$$

where only the third derivatives of $\delta g_{\alpha\beta}$ have been retained. Now it is clear that there exists no linear combination of J_1 and J_2 for which these leading terms in δJ_1 and δJ_2 can be made to cancel. Therefore J_1 and J_2 cannot be included as surface terms in the action. Now consider $J_3, \ldots, J_9$:

$$\delta J_3 = \int [\tfrac{1}{2} R h^{\alpha\beta} \pm K(K^{\alpha\beta} - K g^{\alpha\beta})] n^\rho \nabla_\rho \, \delta g_{\alpha\beta} \, d\Sigma$$
$$\pm \int K(n^\alpha h^{\beta\sigma} + n^\beta h^{\alpha\sigma} - n^\sigma h^{\alpha\beta}) \nabla_\sigma (n^\tau \nabla_\tau \, \delta g_{\alpha\beta}) \, d\Sigma$$

$$\delta J_4 = \frac{1}{2}\int [R_{\mu\nu} h^{\mu\nu} h^{\alpha\beta} + K^2 n^\alpha n^\beta \pm 2K(K^{\alpha\beta} - K g^{\alpha\beta})] n^\rho \nabla_\rho \, \delta g_{\alpha\beta} \, d\Sigma$$
$$\pm \frac{1}{2}\int K(n^\alpha h^{\beta\sigma} + n^\beta h^{\alpha\sigma} - n^\sigma h^{\alpha\beta}) \nabla_\sigma (n^\tau \nabla_\tau \, \delta g_{\alpha\beta}) \, d\Sigma$$

$$\delta J_5 = \frac{1}{2}\int [R_{\mu\nu} h^{\mu\alpha} h^{\nu\beta} \pm (2K^{\alpha\lambda} K^\beta_\lambda - K K^{\alpha\beta} - K^{\mu\lambda} K_{\mu\lambda} g^{\alpha\beta})] n^\rho \nabla_\rho \, \delta g_{\alpha\beta} \, d\Sigma$$
$$\pm \frac{1}{2}\int [K^{\alpha\sigma} n^\beta + K^{\beta\sigma} n^\alpha - K^{\alpha\beta} n^\sigma] \nabla_\sigma (n^\tau \nabla_\tau \, \delta g_{\alpha\beta}) \, d\Sigma$$

$$\delta J_6 = \frac{1}{2}\int [R_{\mu\nu} h^{\mu\alpha} h^{\nu\beta} \mp R^{\mu\alpha\nu\beta} n_\mu n_\nu \pm (2K^\alpha_\lambda K^{\lambda\beta} - K K^{\alpha\beta} - K^{\mu\lambda} K_{\mu\lambda} h^{\alpha\beta})] n^\rho \nabla_\rho \, \delta g_{\alpha\beta} \, d\Sigma$$

$$\delta J_7 = \frac{3}{2}\int K^2 h^{\alpha\beta} n^\rho \nabla_\rho \, \delta g_{\alpha\beta} \, d\Sigma$$

$$\delta J_8 = \int [\tfrac{1}{2} K^{\mu\lambda} K_{\mu\lambda} h^{\alpha\beta} + K K^{\alpha\beta}] n^\rho \nabla_\rho \, \delta g_{\alpha\beta} \, d\Sigma$$

$$\delta J_9 = \frac{3}{2}\int K^{\alpha\lambda} K^\beta_\lambda n^\rho \nabla_\rho \, \delta g_{\alpha\beta} \, d\Sigma.$$

Now δJ_3, δJ_4 and δJ_5 all contain second derivatives of $\delta g_{\alpha\beta}$ which must cancel mutually. This requires J_3 and J_4 to be present only in the combination $J_3 - 2J_4$ and for J_5 to be absent. Hence $\delta(J_3 - 2J_4)$ and δJ_6 are available to cancel surface terms in δI_1, δI_2 and δI_3 and it is not difficult to verify that this can be done only for the combination $I_1 - 4I_2 + I_3$. The final result is that the only term that can be included in the action is $I_1 - 4I_2 + I_3 - 4J_3 + 8J_4 - 8J_6 \pm \frac{8}{3}J_7 \mp 8J_8 \pm \frac{16}{3}J_9$.

It is of interest to investigate what effect the inclusion of the Gauss–Bonnet invariant in the gravitational action has on black hole evaporation. The Euclideanised action is

$$\hat{I}[g] = -\frac{1}{16\pi G}\int (R - 2\Lambda)\sqrt{g} \, d^4x - \frac{1}{8\pi G}\int (K - K^0)\sqrt{h} \, d^3x$$

$$+\frac{\alpha}{32\pi^2}\int (R^2-4R^{\alpha\beta}R_{\alpha\beta}+R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta})\sqrt{g}\,\mathrm{d}^4x$$

$$-\frac{\alpha}{8\pi^2}\int (RK-2R_{\alpha\beta}Kh^{\alpha\beta}+2R_{\alpha\beta\gamma\delta}K^{\alpha\gamma}j^{\beta\delta}-\tfrac{2}{3}K^3+2KK^{\alpha\beta}K_{\alpha\beta}$$

$$-\tfrac{4}{3}K^{\alpha\beta}K_{\beta\gamma}K^{\gamma}_{\alpha}-M^0)\sqrt{h}\,\mathrm{d}^3x$$

where K^0 and M^0 are constants chosen to make the action vanish when $g_{\alpha\beta}$ is the flat space metric $\eta_{\alpha\beta}$. The constant α is a coupling constant and I have included a factor $(32\pi^2)^{-1}$ in the Gauss–Bonnet term so that the invariant multiplying α is actually the Euler characteristic χ of the space–time. If the manifold has a boundary the formula for the Euler characteristic contains boundary terms and these are given precisely by the final surface integral appearing in $\hat{I}[g]$ (Chern 1945). It is not difficult to evaluate $\hat{I}[g]$ for the Euclideanised Schwarzschild metric with black hole mass M

$$\hat{I}[g]=4\pi M^2+2\alpha.$$

The only difference between this and the standard black hole result is the appearance of the constant term 2α which arises because $\chi=2$ for Euclideanised Schwarzschild. The surface terms proportional to α in $\hat{I}[g]$ do not in fact contribute as they fall off rapidly at large Euclidean distances. If one requires that the black hole action be non-negative, the constant α must be non-negative. This is in fact a fairly reasonable assumption to make as χ is negative only for topologies which deviate significantly from simple connectedness and it is not unnatural to suppose that the gravitational path integral should be dominated by topologies which are (in some sense) close to simply connected ones. Taking $\alpha \geqslant 0$ will then ensure that the Gauss–Bonnet term adds a non-negative quantity to the action.

The partition function Z is given by

$$\ln Z=-\hat{I}=-4\pi M^2-2\alpha$$

and the entropy S is easily seen to be

$$S=4\pi M^2-2\alpha.$$

Now as the evaporation process proceeds, the entropy decreases until it reaches zero when

$$M^2=\alpha/2\pi.$$

Since entropy cannot be negative, the inclusion of the Gauss–Bonnet invariant in the action sets a non-zero lower bound to the mass of the black hole which is of the order of the Planck mass if $\alpha\sim 1$. Although this result is certainly a slight improvement on the conventional picture, in which the final state of the black hole is zero mass and infinite temperature, it does not include the effects of the back reaction which become important at Planck dimensions and which will probably modify the final stages of the evaporation process still further.

In this Letter I have shown that the only quadratic curvature scalar that can be included in the gravitational action is the Gauss–Bonnet invariant. The possiblity of including such a term to assist in renormalisation has been considered previously in the context of supergravity by Christensen and Duff (1978, 1979). One possible benefit of including this extra term in the action is that it may rescue 't Hooft and Veltman's (1974) result that pure gravity is one-loop renormalisable which has recently been shown to be in error (Capper and Kimber 1980). Two possible difficulties that need to be overcome if this is to be achieved are (i) the field equations for $n \neq 4$ are no longer $R_{\mu\nu} = 0$ and (ii) there may be new contributions to the one-loop divergences from the Gauss–Bonnet term itself. Further investigation of these matters would be of interest.

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